ABSOLUTE, GORENSTEIN, AND TATE TORSION MODULES

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ABSTRACT. We show that there is an Avramov-Martsinkovsky type exact sequence with \( \hat{\text{Tor}} \), \( \text{Gtor} \), and \( \text{Tor} \). We prove that if \( R \) is a Gorenstein ring then the modules \( \hat{\text{Tor}}^R_n(M, N) \), \( n \geq 1 \) can be computed using either a complete resolution of \( M_R \) or using a complete resolution of \( R_N \). We show that over a Gorenstein ring a left \( R \)-module \( N \) is Gorenstein flat if and only if \( \text{Gtor}^R_1(\cdot, N) = 0 \). We also show that over commutative Gorenstein rings the modules \( \hat{\text{Tor}}^R_n(M, \cdot) \) can be computed by the combined use of a flat resolution and a Gorenstein flat resolution of \( M \).

1. INTRODUCTION

In this article we continue the development of the branch of relative homological algebra that is called Gorenstein homological algebra, but with an emphasis on the part that deals with the derived functors of the tensor product functor.

Our main results are first to show the existence of exact sequences connecting the absolute torsion functors, the Gorenstein torsion functors and the Tate torsion functors. These exact sequences are like those of Avramov and Martsinkovsky in [1] for the extension functors. But we use a completely different approach from theirs (which was restricted to finitely generated modules).

The material presented here deals with several relative derived functors:

- \( \text{Gtor}^R_n(M, -) \) defined via a Gorenstein projective resolution of \( M \)
- \( \text{Gtor}^R_n(M, -) \) defined via a Gorenstein flat resolution of \( M \)
- \( \hat{\text{Tor}}^R_n(M, -) \) defined via a complete resolution of \( M \)
- \( \hat{\text{Tor}}^R_n(g, P)(M, -) = H_{n+1}(M(u) \otimes_R -) \) where \( M(u) \) is the mapping cone of a chain map \( u : P \rightarrow G \). induced by \( \text{id}_M \), with \( P \).

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a deleted projective resolution and $\mathbf{G}$, a deleted Gorenstein projective resolution of $M$ (see Section 2 for definitions).

- $\widetilde{\text{tor}}^R_n(M, -) = H_{n+1}(M(v) \otimes_R -)$ with $M(v)$ the mapping cone of a chain map $v : \mathbf{F} \to \mathbf{D}$, induced by $\text{id}_M$, where $\mathbf{F}$ is a deleted flat resolution and $\mathbf{D}$ is a deleted Gorenstein flat resolution of $M$.

- $\widetilde{\text{Tor}}^R_n(-, N)$ defined via a complete resolution of $N$.

We consider a two-sided noetherian ring $R$. We prove first (Proposition 1) that when $M$ has finite Gorenstein projective dimension we have $\tilde{\text{Tor}}^R(M, -) \simeq \tilde{\text{Tor}}^R_n(M, -)$ for any $n \geq 1$. Using this we show that for each left $R$-module $N$ there is an exact sequence:

\begin{equation}
\ldots \to G\text{tor}^R_2(M, N) \to \tilde{\text{Tor}}^R_1(M, N) \to \text{Tor}^R_1(M, N) \to G\text{tor}^R_1(M, N) \to 0
\end{equation}

If on the other hand $N$ has finite Gorenstein projective dimension then a similar procedure gives an exact sequence:

\begin{equation}
\ldots \to H_2(M \otimes_R \mathbf{G}.) \to \tilde{\text{Tor}}^R_1(M, N) \to \text{Tor}^R_1(M, N) \to H_1(M \otimes_R \mathbf{G}.) \to 0
\end{equation}

where $\mathbf{G}.$ is now a deleted Gorenstein projective resolution of $N$.

Over a Gorenstein ring every module has finite Gorenstein projective dimension, so we have both exact sequences. Also in this situation the same Gorenstein derived functors of $- \otimes_R -$ for given $M, N$ can be computed using either a Gorenstein projective resolution of $M$ or a Gorenstein projective resolution of $N$ i.e.:

\begin{equation}
H_n(M \otimes_R \mathbf{G}.) = G\text{tor}^R_n(M, N), \text{ for any } n \geq 0
\end{equation}

([2], Theorem 12.2.2)

Using (3) and comparing the exact sequences (1) and (2) it is natural to ask if $\tilde{\text{Tor}}^R_n(M, N) \simeq \tilde{\text{Tor}}^R_n(M, N)$. We prove (Theorem 2) that this is true, i.e. the Tate torsion functors $\tilde{\text{Tor}}^R_n(M, N)$ can be computed using either a complete resolution of $M$ or using a complete resolution of $N$. Then we use balancedness of $\tilde{\text{Tor}}$ to show that $\tilde{\text{Tor}}$ commutes with direct limits (Propositions 4 and 5).

We also use the exact sequence (1) to show that over Gorenstein rings, Gorenstein flat modules can be defined in terms of the vanishing of Gorenstein torsion functors (Proposition 2). This result seems to add
evidence that the somewhat mysterious definition of Gorenstein flat modules is the correct one.

In Section 5 we prove that over a commutative Gorenstein ring the Tate torsion functors $\hat{\Tor}_n^R(M, N)$, $n \geq 1$ can also be computed by the combined use of a flat and a Gorenstein flat resolution of $M$. More precisely we show (Proposition 10) that $\hat{\Tor}_n^R(M, N) \simeq \hat{\Tor}_n^R(M, N)$ for any $R$-modules $M, N$, for any $n \geq 1$.

2. Preliminaries

Let $R$ be an associative ring with 1.

We recall first the definition of Gorenstein projective modules.

**Definition 1** ([2], Def. 10.2.1). An $R$-module $M$ is said to be Gorenstein projective if there is a $\Hom(-, \Proj)$ exact exact sequence
\[ \ldots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \ldots \]
of projective modules such that $M = \Ker(P^0 \rightarrow P^1)$.

The Gorenstein flat modules are defined by the tensor product functor $- \otimes_R -$.

**Definition 2** ([2], Def. 10.3.1). A left $R$-module $N$ is said to be Gorenstein flat if there exists an $\Inj \otimes_R -$ exact exact sequence
\[ \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \ldots \]
of flat left $R$-modules such that $N = \Ker(F^0 \rightarrow F^1)$.

**Definition 3** ([2], pp. 167). Let $\mathcal{P}$ be a class of $R$-modules. For an $R$-module $M$ a morphism $\phi : P \rightarrow M$ where $P \in \mathcal{P}$ is a $\mathcal{P}$-precover of $M$ if $\Hom(P', P) \rightarrow \Hom(P', M) \rightarrow 0$ is exact for any $P' \in \mathcal{P}$.

If moreover, any morphism $f : P \rightarrow P$ such that $\phi = \phi \circ f$ is an automorphism of $P$ then $\phi : P \rightarrow M$ is called a $\mathcal{P}$-cover of $M$.

**Definition 4** ([2], Def. 8.1.2). A $\mathcal{P}$-resolution of $M$ is a $\Hom(\mathcal{P}, -)$ exact complex $P : \ldots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ (not necessarily exact) with each $P_i \in \mathcal{P}$.

We note that $P$ is a $\mathcal{P}$-resolution of $M$ if and only if $P_0 \rightarrow M$, $P_1 \rightarrow \Ker(P_0 \rightarrow M)$, and $P_i \rightarrow \Ker(P_{i-1} \rightarrow P_{i-2})$ for $i \geq 2$ are $\mathcal{P}$-precovers.

**Definition 5** ([2], pp. 169). A $\mathcal{P}$-resolution $P : \ldots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ such that $P_0 \rightarrow M$, $P_1 \rightarrow \Ker(P_0 \rightarrow M)$, and $P_i \rightarrow \Ker(P_{i-1} \rightarrow P_{i-2})$ for $i \geq 2$ are $\mathcal{P}$-covers is called minimal.
If $\mathcal{P}$ is a class of $R$-modules that contains all the projective $R$-modules then any $\mathcal{P}$-precover is a surjective map. Therefore any $\mathcal{P}$ resolution of $M$ is an exact complex in this case.

If $\mathbf{P} : \cdots \to P_1 \to P_0 \to M \to 0$ is a $\mathcal{P}$ resolution of $M$ then we refer to the complex $\mathbf{P} : \cdots \to P_1 \to P_0 \to 0$ as a deleted resolution of $M$. A $\mathcal{P}$-resolution of an $R$-module $M$ is unique up to homotopy ([2], pg. 169) so it can be used to compute derived functors.

By a Gorenstein projective resolution of $M$ we mean a resolution in the sense of Definition 4 when $\mathcal{P}$ is the class of Gorenstein projective modules.

**Definition 6 ([3], pp. 1920).** Let $M$ be a right $R$-module that has a Gorenstein projective resolution $\mathbf{G}$. Then $\text{Gtor}_n^R(M, N) = H_n(\mathbf{G.} \otimes_R N)$ for each left $R$-module $N$, for any $n \geq 0$, where $\mathbf{G.}$ is the deleted Gorenstein projective resolution of $M$.

We can also compute left derived functors of $M \otimes_R N$ using resolutions by Gorenstein flat modules. We denote these $\text{gtor}_i^R(M, N)$ to distinguish them from the functors $\text{Gtor}_i^R(M, N)$.

**Definition 7 ([2], pp. 299).** If $M$ is a right $R$-module that has a Gorenstein flat resolution $\mathbf{F}$ then $\text{gtor}_n^R(M, N) = H_n(\mathbf{F.} \otimes_R N)$ for each left $R$-module $N$, for each $n \geq 0$. ($\mathbf{F.}$ is the deleted Gorenstein flat resolution of $M$).

A ring $R$ is said to be Gorenstein if it is both left and right noetherian and has finite self injective dimension on both sides. Over a Gorenstein ring every module has a finite Gorenstein projective resolution and a finite Gorenstein flat resolution ([2], Theorem 11.5.1 and Theorem 11.7.3).

By [2] Theorem 12.2.2, if $R$ is a Gorenstein ring then the modules $\text{Gtor}_n^R(M, N)$ can also be computed using a Gorenstein projective resolution of the left $R$-module $N$.

Again by [2], Theorem 12.2.2, if $R$ is a Gorenstein ring then for each $n \geq 0$ $\text{gtor}_n^R(M, N) = H_n(M \otimes_R \mathbf{V.})$ where $\mathbf{V.}$ is a deleted Gorenstein flat resolution of $N$.

If $R$ is Gorenstein then $\text{gtor}_n^R(-, -) \cong \text{Gtor}_n^R(-, -)$ for any $n \geq 0$ ([2], pg. 299).

**Remark 1.** H. Holm studied the Gorenstein torsion functors over arbitrary rings. For more results on $\text{Gtor}$ and $\text{gtor}$ see [3].
3. An Avramov-Martsinkovsky type exact sequence with \( \widehat{\text{Tor}}, \text{Tor} \) and \( \text{Gtor} \)

Let \( R \) be a two-sided noetherian ring and let \( M \) be a right \( R \)-module that has a Gorenstein projective resolution. Such a resolution can be used to compute left derived functors \( \text{Gtor}^R_i(M, N) \). There are obvious natural maps \( \text{Tor}^R_i(M, N) \to \text{Gtor}^R_i(M, N) \) for all \( i \geq 0 \), and \( \text{Tor}^R_0(M, N) \simeq \text{Gtor}^R_0(M, N) \).

The main result of this section is showing the existence of an exact sequence:

\[
\ldots \to \text{Gtor}^R_2(M, N) \to \text{Tor}^R_1(M, N) \to \text{Tor}^R_1(M, N) \to \text{Gtor}^R_1(M, N) \to 0
\]

when \( M \) has finite Gorenstein projective dimension.

This comes to show that the Tate Tor (\( \widehat{\text{Tor}} \)) measures the “difference” between the absolute Tor and the Gorenstein Tor (\( \text{Gtor} \)). In particular it shows that \( \text{Tor}^R_i(M, -) \to \text{Gtor}^R_i(M, -) \) is an isomorphism for all \( i \geq 1 \) if and only if \( \text{Tor}^R_i(M, -) = 0 \) for all \( i \geq 1 \).

We recall first the following:

**Definition 8.** A complex \( T \) is totally acyclic if it is exact, each module of \( T \) is projective, and \( \text{Hom}(T, Q) \) is exact for any projective \( R \)-module \( Q \).

The Tate torsion functors are defined by means of a complete resolution of \( M \):

**Definition 9** ([1]). A complete resolution of an \( R \)-module \( M \) is a diagram \( T \xrightarrow{u} P \to M \) where \( P \to M \) is a deleted projective resolution of \( M \), \( T \) is a totally acyclic complex, \( u \) is a morphism of complexes and \( u_n \) is bijective for all \( n \gg 0 \).

If \( M \) has such a complete resolution \( T \xrightarrow{u} P \to M \) then for each left \( R \)-module \( N \), \( \text{Tor}_n^R(M, N) = H_n(T \otimes_R N) \), for any \( n \in \mathbb{Z} \).

**Definition 10** ([2], Definition 4.1). A module \( M \) has finite Gorenstein projective dimension if there is a Gorenstein projective resolution of \( M \) of the form

\[
0 \to G_n \to G_{n-1} \to \ldots \to G_0 \to M \to 0
\]

If \( n \) is the least with this property then we set \( \text{Gor proj dim } M = n \).
We show first (Proposition 1) that if $M$ is a right $R$-module with $\text{Gor proj dim } M < \infty$ then the modules $\widehat{\text{Tor}}_n^R(M, N)$, $n \geq 1$ can also be computed by the combined use of a projective and a Gorenstein projective resolution of $M$.

Let $P$ be a projective resolution and $G$ be a Gorenstein projective resolution of $M$. Let $u : P \to G$ be a map of complexes induced by $\text{id}_M$ and let $M(u)$ be its mapping cone. For each left $R$-module $N$, we define $\widehat{\text{Tor}}_n^g,P(M, N)$ by the equality

$$\widehat{\text{Tor}}_n^g,P(M, N) = H_{n+1}(M(u) \otimes_R N),$$

for any $n \geq 1$.

We showed ([6], pp. 392) that if $P, P'$ are two projective resolutions of $M$, $G, G'$ are two Gorenstein projective resolutions of $M$, $u : P \to G$ and $v : P' \to G'$. are two maps of complexes induced by $\text{id}_M$ then $M(u) \sim M(v)$. So $\widehat{\text{Tor}}_n^g,P(M, -)$ is well defined.

**Proposition 1.** If $\text{Gor proj dim } M < \infty$ then $\widehat{\text{Tor}}_n^R(M, N) \cong \widehat{\text{Tor}}_n^g,P(M, N)$ for any left $R$-module $N$, for any $n \geq 1$.

**Proof.** Let $\ldots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ be a projective resolution of $M$.

Since $\text{Gor proj dim } M = g < \infty$ it follows that $C = \text{Ker } f_{g-1}$ is Gorenstein projective ([4], Theorem 2.20). Thus there is a $\text{Hom}( -, \text{Proj})$ exact exact complex $T$,

$$T = \ldots \to T_{g+2} \xrightarrow{d_{g+2}} T_{g+1} \xrightarrow{d_{g+1}} T_g \xrightarrow{d_g} T_{g-1} \to \ldots$$

with each $T_n$ projective, such that $C = \text{Im } d_g$.

Then

$$P = \ldots \to T_{g+1} \xrightarrow{d_{g+1}} T_g \xrightarrow{d_g} P_{g-1} \xrightarrow{f_{g-1}} \ldots \to P_1 \to P_0 \to M \to 0$$

is a projective resolution of $M$.

Since $\text{Hom}(T, P_j)$, $j \geq 0$ is exact there are homomorphisms $u_0, u_1, \ldots, u_{g-1}$ that make the diagram commutative:

$$\begin{array}{ccc}
T : & \ldots & \to T_{g+1} \xrightarrow{d_{g+1}} T_g \xrightarrow{d_g} T_{g-1} \xrightarrow{d_{g-1}} \ldots \\
& | & \downarrow \quad \text{Id} \quad \downarrow \quad u_{g-1} \\
& | & \downarrow \\
P : & \ldots & \to T_{g+1} \xrightarrow{d_{g+1}} T_g \xrightarrow{d_g} P_{g-1} \xrightarrow{f_{g-1}} \ldots
\end{array}$$
is a complete resolution of $M$.

Let $D = \text{Im} d_0$. Then $D$ is Gorenstein projective and there is a commutative diagram

$$
\begin{array}{cccccccc}
0 & \longrightarrow & C & \longrightarrow & T_{g-1} & \longrightarrow & T_{g-2} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow u_{g-1} & & \downarrow u_{g-2} & \\
0 & \longrightarrow & C & \longrightarrow & P_{g-1} & \longrightarrow & P_{g-2} & \longrightarrow & \cdots \\
\end{array}
$$

with both rows exact complexes. Consequently the mapping cone is an exact complex.

The mapping cone has the exact subcomplex $0 \rightarrow C \rightarrow C \rightarrow 0$. Forming the quotient we get an exact complex:

$$
G = 0 \rightarrow T_{g-1} \rightarrow P_{g-1} \oplus T_{g-2} \rightarrow \cdots \rightarrow P_1 \oplus T_0 \rightarrow P_0 \oplus D \rightarrow M \rightarrow 0
$$

Since $K = \text{Ker}(P_0 \oplus D \rightarrow M)$ has finite projective dimension it follows that $G$ is a Gorenstein projective resolution of $M$.

There is a map of complexes $e : P \rightarrow G$

$$
\begin{array}{cccccccc}
\cdots & \longrightarrow & T_{g+1} & \longrightarrow & T_g & \longrightarrow & T_{g-1} & \longrightarrow & \cdots \\
\downarrow d_{g+1} & & \downarrow d_g & & \downarrow d_g & & \downarrow \ldots \\
\cdots & \longrightarrow & 0 & \longrightarrow & T_{g-1} & \longrightarrow & P_{g-1} \oplus T_{g-2} & \longrightarrow & \cdots \\
\end{array}
$$
for any \( H \)

We have

\[
\alpha_j : T_j \to P_{j+1} \oplus T_j \oplus P_j.
\]

\( \alpha_j(x) = (0, x, -u_j(x)) \) for any \( x \in T_j, 1 \leq j \leq g - 1 \).

\( \alpha_{g-1} : T_{g-1} \to T_{g-1} \oplus P_{g-1}, \alpha_{g-1}(x) = (x, -u_{g-1}(x)), \forall x \in T_{g-1}; \)

\( \alpha_{g+j} = -id_{T_{g+j}} \) if \( j \geq 1 \) is odd; \( \alpha_{g+j} = id_{T_{g+j}} \) if \( j \geq 0 \) is even.

There is also a map of complexes \( l : M(e) \to \overline{T}[1] : \)

\( l' : P_0 \oplus D \to D, l'(x, y) = y, \forall (x, y) \in P_0 \oplus D. \)

\( l_j : P_{j+1} \oplus T_j \oplus P_j \to T_j, l_j(x, y, z) = y, \forall (x, y, z) \in P_{j+1} \oplus T_j \oplus P_j \)

\( l_{g-1} : T_{g-1} \oplus P_{g-1} \to T_{g-1}, l_{g-1}(x, y) = x, \forall (x, y) \in T_{g-1} \oplus P_{g-1}. \)

\( l_{g+j} = -id_{T_{g+j}} \) if \( j \geq 1 \) is odd; \( l_{g+j} = id_{T_{g+j}} \) if \( j \geq 0 \) is even.

We have \( l \circ \alpha = id_{\overline{T}[1]} \) and \( \alpha \circ l \sim id_{M(e)}. \)

(A chain homotopy between \( \alpha \circ l \) and \( Id_{M} \) is given by the maps:

\( \chi' : P_0 \oplus D \to P_1 \oplus T_0 \oplus P_0, \chi'(x, y) = (0,0, -x) \)

\( \chi_j : P_{j+1} \oplus T_j \oplus P_j \to P_{j+2} \oplus T_{j+1} \oplus P_{j+1}, \chi_j(x, y, z) = (0,0, -x), \)

\( 1 \leq j \leq g - 3 \)

\( \chi_{g-2} : P_{g-1} \oplus T_{g-2} \oplus P_{g-2} \to T_{g-1} \oplus P_{g-1}, \chi_{g-2}(x, y, z) = (0, -x). \)

So \( H_{n+1}(M(e) \otimes R N) \simeq H_{n+1}((T)[1] \otimes R N) \leftrightarrow \overline{Tor}_n^{G,P}(M, N) \simeq \overline{Tor}_n^R(M, N), \)

for any \( R N, \) for any \( n \geq 1. \)
We can prove now the existence of an Avramov-Martsinkovsky type exact sequence with $\hat{\text{Tor}}, \text{Tor}$, and $\text{Gtor}$.

**Theorem 1.** Let $M$ be a right $R$-module with $\text{Gor} \text{ proj dim } M < \infty$. For each left $R$-module $N$ there is an exact sequence:

$$
\ldots \rightarrow \text{Gtor}_2^R(M, N) \rightarrow \hat{\text{Tor}}_1^R(M, N) \rightarrow \text{Tor}_1^R(M, N) \rightarrow \text{Gtor}_1^R(M, N) \rightarrow 0
$$

**Proof.** Let $P$ be a projective resolution of $M$, $G$ be a Gorenstein projective resolution of $M$ and let $u : P \rightarrow G$ be a chain map induced by $\text{id}_M$.

\[
\begin{array}{ccccccc}
P : \ldots & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & M & \rightarrow & 0 \\
 & & \uparrow u_1 & & \uparrow u_0 & & \\
G : \ldots & \rightarrow & G_1 & \rightarrow & G_0 & \rightarrow & M & \rightarrow & 0
\end{array}
\]

Both $P$ and $G$ are exact complexes so the mapping cone $\overline{\text{M}(u)} : \ldots \rightarrow G_2 \oplus P_1 \rightarrow G_1 \oplus P_0 \rightarrow G_0 \oplus M \rightarrow M \rightarrow 0$ is also exact. $\overline{\text{M}(u)}$ has the exact subcomplex $\overline{M} : 0 \rightarrow M \xrightarrow{\text{id}} M \rightarrow 0$. Forming the quotient we obtain the mapping cone $\overline{\text{M}(u)}$ of $u : P \rightarrow G$. Thus $\overline{\text{M}(u)}$ is exact.

The sequence $0 \rightarrow \text{G.} \rightarrow \text{M}(u) \rightarrow P \cdot [1] \rightarrow 0$ is split exact in each degree so for each left $R$-module $N$ we have an exact sequence of complexes $0 \rightarrow \text{G.} \otimes_R N \rightarrow \text{M}(u) \otimes_R N \rightarrow P \cdot [1] \otimes_R N \rightarrow 0$. Therefore we have a long exact sequence:

$$
\ldots \rightarrow H_{n+1}(\text{G.} \otimes_R N) \rightarrow H_{n+1}(\text{M}(u) \otimes_R N) \rightarrow H_{n+1}(P \cdot [1] \otimes_R N) \\
\rightarrow H_n(\text{G.} \otimes_R N) \rightarrow H_n(\text{M}(u) \otimes_R N) \rightarrow \ldots
$$

Since $\text{M}(u)$ is exact and the functor $- \otimes_R N$ is right exact it follows that $H_1(\text{M}(u) \otimes_R N) = H_0(\text{M}(u) \otimes_R N) = 0$.

So the exact sequence above is:

$$
\ldots \rightarrow H_2(\text{G.} \otimes_R N) \rightarrow H_2(\text{M}(u) \otimes_R N) \rightarrow H_2(P \cdot [1] \otimes_R N) \\
\rightarrow H_1(\text{G.} \otimes_R N) \rightarrow 0 \rightarrow H_1(P \cdot [1] \otimes_R N) \rightarrow H_0(\text{G.} \otimes_R N) \rightarrow 0
$$

This gives us the exact sequence

$$
(4) \quad \ldots \rightarrow \text{Gtor}_2^R(M, N) \rightarrow \hat{\text{Tor}}_1^G(M, N) \rightarrow \text{Tor}_1^R(M, N) \rightarrow \text{Gtor}_1^R(M, N) \rightarrow 0
$$
By Proposition 1 we have \( \hat{\text{Tor}}_n^G(M, N) \simeq \hat{\text{Tor}}_n^R(M, N) \), for any \( n \geq 1 \).
So we obtain the desired long exact sequence. \( \square \)

**Corollary 1.** Let \( R \) be a Gorenstein ring. The following are equivalent for a right \( R \)-module \( M \):

1. \( \text{proj dim } M < \infty \)
2. \( \hat{\text{Tor}}_n^R(M, -) = 0 \) for all \( n \in \mathbb{Z} \)

**Proof.** 1) \( \Rightarrow \) 2) If \( \text{proj dim } M < \infty \) then a complete resolution of \( M \) is \( 0 \rightarrow P \rightarrow M \) where \( P \) is a bounded projective resolution of \( M \). So \( \hat{\text{Tor}}_n^R(M, -) = 0 \) for any \( n \in \mathbb{Z} \).

2) \( \Rightarrow \) 1) Since \( R \) is Gorenstein we have \( \text{Gor proj dim } M < \infty \). The exact sequence

\[
\cdots \rightarrow \text{Gtor}_2^R(M, N) \rightarrow \hat{\text{Tor}}_1^R(M, N) \rightarrow \text{Tor}_1^R(M, N) \\
\rightarrow \text{Gtor}_1^R(M, N) \rightarrow 0
\]

gives that \( \text{Tor}_n^R(M, -) \rightarrow \text{Gtor}_n^R(M, -) \) is an isomorphism for all \( n \geq 1 \). By [2], Proposition 12.3.3, \( \text{proj dim } M < \infty \).

We use Theorem 1 to show that over Gorenstein rings, Gorenstein flat modules can be characterized in terms of the vanishing of the Gorenstein torsion functors.

**Proposition 2.** Let \( R \) be a Gorenstein ring. For a left \( R \)-module \( N \) the following are equivalent:

1. \( N \) is Gorenstein flat.
2. \( \text{Gtor}_n^R(-, N) = 0 \), for any \( n \geq 1 \).
3. \( \text{Gtor}_1^R(-, N) = 0 \).

**Proof.** 1) \( \Rightarrow \) 2) \( N \) is a Gorenstein flat left \( R \)-module so \( N \simeq \lim \rightarrow C_i \) for some inductive system \( ((C_i), (f_{ji})) \) with each \( C_i \) a finitely generated Gorenstein projective left \( R \)-module ([2], Theorem 10.3.8(4)).

Let \( M \) be any right \( R \)-module and let \( G \) be a deleted Gorenstein projective resolution of \( M \). We have \( G \otimes_R (\lim \rightarrow C_i) \simeq \lim \rightarrow (G \otimes_R C_i) \). Then \( H_n(G \otimes_R (\lim \rightarrow C_i)) \simeq H_n(\lim \rightarrow (G \otimes_R C_i)) \simeq \lim \rightarrow (H_n(G \otimes_R C_i)) \simeq 0 \iff \text{Gtor}_n^R(M, N) = 0 \) for any \( n \geq 1 \).

2) \( \Rightarrow \) 3) Straightforward.
3) ⇒ 1) Let \( L \) be a right \( R \)-module with \( \text{proj dim } L < \infty \). By Corollary 1 \( \widehat{\text{Tor}}_n^R(L, N) = 0 \), for any \( n \geq 1 \). The exact sequence

\[
\ldots \rightarrow \text{Tor}_2^R(L, N) \rightarrow \text{Gtor}_2^R(L, N) \rightarrow \widehat{\text{Tor}}_1^R(L, N) \rightarrow \text{Tor}_1^R(L, N) \rightarrow \text{Gtor}_1^R(L, N) \rightarrow 0
\]

gives us \( \text{Tor}_1^R(L, N) \simeq \text{Gtor}_1^R(L, N) = 0 \) (by hypothesis).

Since \( \text{Tor}_1^R(L, N) = 0 \) for any \( L_R \) with \( \text{proj dim } L < \infty \) it follows that \( N \) is Gorenstein flat (by [2], Theorem 10.3.8).

\[ \square \]

**Theorem 2.** Let \( R \) be a Gorenstein ring and let \( N \) be a left \( R \)-module. The following are equivalent:

1. \( \text{Gor flat dim } N \leq r \)
2. \( \text{Gtor}_i^R(M, N) = 0 \) for any right \( R \)-module \( M \), for any \( i \geq r + 1 \).

**Proof.** 1) ⇒ 2) Since \( \text{Gor flat dim } N \leq r \) there is a Gorenstein flat resolution of \( N \), \( F : 0 \rightarrow F_r \rightarrow \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0 \). Then the complex \( M \otimes_R F \) has length \( r \) and we have \( \text{Gtor}_i^R(M, N) \simeq \text{gtor}_i^R(M, N) = H_i(M \otimes_R F) = 0 \) for \( i > r \).

2) ⇒ 1) If \( L \) is a right \( R \)-module with \( \text{proj dim } L < \infty \) then \( \widehat{\text{Tor}}_n^R(L, -) = 0 \) for any \( n \in \mathbb{Z} \). The exact sequence

\[
\ldots \rightarrow \text{Gtor}_{r+2}^R(L, N) \rightarrow \widehat{\text{Tor}}_{r+1}^R(L, N) \rightarrow \text{Tor}_{r+1}^R(L, N) \rightarrow \text{Gtor}_r^R(L, N) \rightarrow \widehat{\text{Tor}}_r^R(L, N) \rightarrow \ldots
\]

gives \( \text{Tor}_i^R(L, N) = 0 \) for any \( i \geq r + 1 \), for any \( L \) with \( \text{proj dim } L < \infty \).

By [2], Proposition 11.7.5, \( \text{Gor flat dim } N \leq r \). \[ \square \]

So over Gorenstein rings,

\( \text{Gor flat dim } N = \text{Sup}\{i \in \mathbb{N}_0 | \text{Gtor}_i^R(M, N) \neq 0 \text{ for some } M_R\} \)

**Proposition 3.** If \( R \) is a Gorenstein ring then \( \text{Gor flat dim } N \leq \text{Gor proj dim } N \) for any left \( R \)-module \( N \).

**Proof.** Since \( R \) is Gorenstein we have \( \text{Gor proj dim } N = r < \infty \).

So \( N \) has a Gorenstein projective resolution.

\[
0 \rightarrow G_r \rightarrow G_{r-1} \rightarrow \ldots \rightarrow G_1 \rightarrow G_0 \rightarrow N \rightarrow 0
\]

Then \( \text{Gtor}_{r+i}^R(M, N) = 0 \) for any \( M_R \), for any \( i \geq 1 \). Thus \( \text{Gor flat dim } N \leq r \). \[ \square \]
4. Balance of $\widehat{\text{Tor}}$

$R$ is a left and right noetherian ring.

Let $N$ be a left $R$-module with $\text{Gor proj dim } N < \infty$. Then we can define Tate torsion functors $\widehat{\text{Tor}}^R_n(-, N)$ by means of a complete resolution of $N$: if $V \to P \to N$ is a complete resolution of $N$ then for each $M_R$ and for each $n \in \mathbb{Z}$ let $\widehat{\text{Tor}}^R_n(M, N) = H_n(M \otimes_R V)$.

The modules $\widehat{\text{Tor}}^R_n(M, N)$, $n \geq 1$ can also be computed by the combined use of a projective and a Gorenstein projective resolution of $N$. More precisely: if $P_\cdot$ and $G_\cdot$ are deleted projective and respectively Gorenstein projective resolutions of $N$ and $v : P_\cdot \to G_\cdot$ is a map of complexes induced by $\text{id}_N$ then a similar argument to the proof of Proposition 1 gives us $\widehat{\text{Tor}}^R_n(M, N) = H_{n+1}(M \otimes_R M(v))$ for any right $R$-module $M$, for any $n \geq 1$.

By arguments similar to those in Section 3 (proof of Theorem 1) we obtain the exact sequence:

$$\ldots \to H_2(M \otimes_R G_\cdot) \to \widehat{\text{Tor}}^R_1(M, N) \to \text{Tor}_1^R(M, N) \to H_1(M \otimes_R G_\cdot) \to 0$$

If $R$ is Gorenstein then $H_n(M \otimes_R G_\cdot) = \text{Gtor}_n^R(M, N)$, for any $n \geq 1$ ([2], Theorem 12.2.2).

Thus for a Gorenstein ring $R$ we have the exact sequence

$$\ldots \to \text{Gtor}_2^R(M, N) \to \widehat{\text{Tor}}^R_1(M, N) \to \text{Tor}_1^R(M, N) \to \text{Gtor}_1^R(M, N) \to 0$$

By comparing the exact sequences (1) and (5) it is natural to ask if, over Gorenstein rings, we have $\widehat{\text{Tor}}^R_n(-, -) \simeq \widehat{\text{Tor}}^R_n(-, -)$ for any $n \geq 1$. We show (Theorem 3) that this is true.

**Theorem 3.** If $R$ is a Gorenstein ring then $\widehat{\text{Tor}}^R_n(M, N) \simeq \widehat{\text{Tor}}^R_n(M, N)$ for any right $R$-module $M$, any left $R$-module $N$, for all $n \in \mathbb{Z}$.

**Proof.** - We prove first that $\widehat{\text{Tor}}^R_n(M, N) \simeq \widehat{\text{Tor}}^R_n(M, N)$ for all $M_R$, $R N$, and for all $n \geq 1$.

Let $g = \text{Gor proj dim } M$. $R$ is Gorenstein, so $g < \infty$.

- Case $g = 0$
Since $\text{Gtor}_i^R(M, -) = 0$ for any $i \geq 1$, the exact sequence

\[ \ldots \to \text{Gtor}_2^R(M, -) \to \text{Tor}_1^R(M, -) \to \text{Gtor}_1^R(M, -) \to \text{Gtor}_1^R(M, -) = 0 \]

gives us $\text{Tor}_i^R(M, N) \simeq \text{Tor}_i^R(M, N)$ for any left $R$-module $N$.

$R$ is Gorenstein so we also have the exact sequence:

\[ \ldots \to \text{Gtor}_2^R(M, N) \to \text{Tor}_1^R(M, N) \to \text{Gtor}_1^R(M, N) \to \text{Gtor}_1^R(M, N) = 0 \]

with $\text{Gtor}_1^R(M, -) = 0, \forall i \geq 1$. It follows that $\text{Tor}_i^R(M, N)$, for any $R$-$N$, for any $i \geq 1$.

Hence $\text{Tor}_i^R(M, N) \simeq \text{Tor}_i^R(M, N) \simeq \text{Tor}_i^R(M, N)$ for any $R$-$N$, for any $i \geq 1$ (similarly, $\text{Tor}_i^R(M, N) \simeq \text{Tor}_i^R(M, N)$ if $R$-$N$ Gorenstein projective).

• Case $g \geq 1$

$R$ is Gorenstein so there is an exact sequence $0 \to M \to L \to C \to 0$ with $C$ a Gorenstein projective right $R$-module and with $\text{proj dim } L < \infty$ ([2], Remark 11.5.10).

Let $N$ be any left $R$-module and let $V$ be a complete resolution of $N$. Each $V_n$ is projective so $0 \to M \otimes_R V \to L \otimes_R V \to C \otimes_R V \to 0$ is an exact sequence of complexes.

We have the long exact sequence:

\[ \ldots \to H_{n+1}(L \otimes_R V) \to H_{n+1}(C \otimes_R V) \to H_n(M \otimes_R V) \to H_n(L \otimes_R V) \to \ldots \]

Since each $K_n = \text{Ker}(V_n \to V_{n-1})$ is a Gorenstein projective left $R$-module and $\text{inj dim } L < \infty$ it follows that $\text{Tor}_1(L, K_n) = 0$ ([2], Theorem 10.3.8) $\forall n \in \mathbb{Z}$. Therefore $L \otimes_R V$ is an exact complex. So

(6) $H_n(M \otimes_R V) \simeq H_{n+1}(C \otimes_R V) \Leftrightarrow \text{Tor}_n^R(M, N) \simeq \text{Tor}_{n+1}^R(C, N)$

for any $n \in \mathbb{Z}$, for any $R$-$N$.

Since $R$ is Gorenstein, for any $R$-$N$ there is an exact sequence $0 \to L' \to C' \to N \to 0$ with $C'$ a Gorenstein projective left $R$-module and with $\text{proj dim } L' < \infty$.

If $M_R$ is any right $R$-module and $T$ is a complete resolution of $M$ then $0 \to T \otimes_R L' \to T \otimes_R C' \to T \otimes_R N' \to 0$ is an exact sequence of complexes. Therefore we have a long exact sequence:

\[ \ldots \to H_{n+1}(T \otimes_R L') \to H_{n+1}(T \otimes_R C') \to H_{n+1}(T \otimes_R N') \to H_n(T \otimes_R L') \to \ldots \]
Since $T \otimes_R L'$ is an exact complex it follows that

\[ H_n(T \otimes_R C') \simeq H_n(T \otimes_R N') \iff \hat{\text{Tor}}_n^R(M, N) \simeq \hat{\text{Tor}}_n^R(M, C'), \]

for any $M_R$, for any $n \in \mathbb{Z}$.

By (6) we have $\text{Tor}_i^R(M, N) \simeq \text{Tor}_{i+1}^R(C, N)$ (since $C \in \text{Gor Proj}$). Then by (7), $\hat{\text{Tor}}_{i+1}(C, N) \simeq \hat{\text{Tor}}_{i+1}(C, C')$. So $\text{Tor}_i^R(M, N) \simeq \text{Tor}_{i+1}^R(C, C')$, $\forall i \geq 1$.

By (7), $\hat{\text{Tor}}_i^R(M, N) \simeq \hat{\text{Tor}}_i^R(M, C') \simeq \hat{\text{Tor}}_i^R(M, C)$ since $C' \in \text{Gor Proj}$.

Thus

\[ \text{Tor}_i^R(M, N) \simeq \text{Tor}_i^R(M, N) \text{ for all } M_R, RN \text{ and all } i \geq 1. \]

**Case** $n = -k$ with $k \geq 0$

We consider again the short exact sequence $0 \rightarrow M \rightarrow L \rightarrow C \rightarrow 0$ with $C$ Gorenstein projective and with $\text{proj dim } L < \infty$.

Let $F = \ldots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \ldots$ be a complete resolution of $C$ ($C = \text{Ker}(F_0 \rightarrow F_{-1})$) and let $C_i = \text{Ker}(F_i \rightarrow F_{i-1})$. If $N$ is any left $R$-module and $V$ is a complete resolution of $N$ then $0 \rightarrow C \otimes_R V \rightarrow F_0 \otimes_R V \rightarrow C_{-1} \otimes_R V \rightarrow 0$ is an exact sequence of complexes and $F_0 \otimes_R V$ is exact (since $F_0$ is projective). The long exact sequence:

\[ \ldots \rightarrow H_{n+1}(F_0 \otimes_R V) \rightarrow H_{n+1}(C_{-1} \otimes_R V) \rightarrow H_n(C \otimes_R V) \rightarrow H_n(F_0 \otimes_R V) \rightarrow \ldots \]

gives us

\[ \text{Tor}_n^R(C, N) \simeq \text{Tor}_{n+1}^R(C_{-1}, N), \text{ for any } n \in \mathbb{Z} \]

Similarly,

\[ \text{Tor}_n^R(C, N) \simeq \text{Tor}_{n+p}^R(C_{-p}, N) \]

By (6) and (9) we have

\[ \text{Tor}_n^R(M, N) \simeq \text{Tor}_{n+p}^R(C_{-p-1}, N). \]

for any $RN$, any $n \in \mathbb{Z}$, and any $p \geq 1$.

$R$ is Gorenstein, so for any $RN$ we have an exact sequence $0 \rightarrow L' \rightarrow C' \rightarrow N \rightarrow 0$ with $C'$ Gorenstein projective and with $\text{proj dim } L' < \infty$. 
Let $G = \ldots \to G_1 \to G_0 \to G_{-1} \to \ldots$ be a complete resolution of $C'$ ($C' = \text{Ker}(G_0 \to G_1)$) and let $C'_i = \text{Ker}(G_i \to G_{i-1})$. Then $C'_i$ is Gorenstein projective for any $i \in \mathbb{Z}$.

A similar argument to the one above gives us:

(11) $\widehat{\text{Tor}}_n^R(M, C') \simeq \widehat{\text{Tor}}_{n+1}^R(M, C'_{-1})$

for any $M_R$ and any $n \in \mathbb{Z}$.

By (7) and (11) we have $\widehat{\text{Tor}}_n^R(M, N) \simeq \widehat{\text{Tor}}_{n+1}^R(M, C'_{-1})$.

Similarly,

(12) $\widehat{\text{Tor}}_n^R(M, N) \simeq \widehat{\text{Tor}}_{n+p}^R(M, C'_{-p})$

for any $M_R$, any $n \in \mathbb{Z}$, and any $p \geq 1$.

By (10), $\overline{\text{Tor}}_{-k}^R(M, N) \simeq \overline{\text{Tor}}_{k}^R(C_{-k-2}, N) \simeq \overline{\text{Tor}}_{k}^R(C_{-k-2}, N)$ (since $C_{-k-2}$ is Gorenstein projective). Then by (12) we have

$\overline{\text{Tor}}_1^R(C_{-k-2}, N) \simeq \overline{\text{Tor}}_{k+2}^R(C_{-k-2}, C'_{-k-1})$

So $\overline{\text{Tor}}_{-k}^R(M, N) \simeq \overline{\text{Tor}}_{k+2}^R(C_{-k-2}, C'_{-k-1})$

By (12), $\overline{\text{Tor}}_{-k}^R(M, N) \simeq \overline{\text{Tor}}_{1}^R(M, C'_{-k-1}) \simeq \overline{\text{Tor}}_{1}^R(M, C'_{-k-1})$ (since $C'_{-k-1}$ is Gorenstein projective). Then by (10) we have $\overline{\text{Tor}}_{1}^R(M, C'_{-k-1}) \simeq \overline{\text{Tor}}_{k+2}^R(C_{-k-2}, C'_{-k-1})$. Thus $\overline{\text{Tor}}_{-k}^R(M, N) \simeq \overline{\text{Tor}}_{k+2}^R(C_{-k-2}, C'_{-k-1})$.

So $\overline{\text{Tor}}_{-k}^R(M, N) \simeq \overline{\text{Tor}}_{k+2}^R(C_{-k-2}, C'_{-k-1}) \simeq \overline{\text{Tor}}_{k+2}^R(C_{-k-2}, C'_{-k-1}) \simeq \overline{\text{Tor}}_{-k}^R(M, N)$

for all $M_R$, $R N$ and all $k \geq 0$.

\[\Box\]

Corollary 2. Let $R$ be a Gorenstein ring. The following are equivalent for a left $R$-module $L$:

1. $\text{proj dim} \ L < \infty$
2. $\overline{\text{Tor}}_n^R(-, L) = 0$ for any $n \in \mathbb{Z}$

Proof. 1) $\Rightarrow$ 2) By Theorem 3, $\overline{\text{Tor}}_n^R(-, L) \simeq \overline{\text{Tor}}_n^R(-, L)$. Since $\text{proj dim} \ L < \infty$ a complete resolution of $L$ is $0 \to \mathbf{P} \to L$ whenever $\mathbf{P}$ is a bounded projective resolution of $L$. So $\overline{\text{Tor}}_n^R(-, L) = 0$ for any $n \in \mathbb{Z}$.
2) ⇒ 1) For each \( N_R \) we have an exact sequence:

\[
\ldots \to G\text{tor}_2^R(N, L) \to \widehat{\text{Tor}}_1^R(N, L) \to \text{Tor}_1^R(N, L) \to G\text{tor}_1^R(N, L) \to 0
\]

Since \( \widehat{\text{Tor}}_n^R(N, L) = 0, \forall n \geq 1 \), it follows that \( G\text{tor}_n^R(N, L) \simeq \text{Tor}_n^R(N, L) \) for any \( n \geq 1 \), for any \( N_R \).

For \( N \in \text{Gor Flat} \) we have \( \text{Tor}_n^R(N, L) \simeq G\text{tor}_n^R(N, L) \simeq g\text{tor}_n^R(N, L) = 0 \) \( \forall n \geq 1 \).

By [2], Proposition 11.5.9, \( \text{proj dim } L < \infty \). □

Using balancedness of \( \widetilde{\text{Tor}} \) we can prove now that over Gorenstein rings \( \widetilde{\text{Tor}} \) commutes with direct limits.

**Proposition 4.** Let \( R \) be a Gorenstein ring. For any left \( R \)-module \( N \) and any inductive system \( ((M_i), (f_{ji})) \) of right \( R \)-modules we have \( \widetilde{\text{Tor}}_n^R(\lim\rightarrow M_i, N) \simeq \lim\rightarrow \widetilde{\text{Tor}}_n^R(M_i, N) \), for any \( n \in \mathbb{Z} \).

**Proof.** Since \( R \) is a Gorenstein ring it follows that \( \text{Gor proj dim } N < \infty \). So \( N \) has a complete resolution \( U \). We have \( (\lim\rightarrow M_i) \otimes_R U \simeq \lim\rightarrow (M_i \otimes_R U) \).

Then \( H_n((\lim\rightarrow M_i) \otimes_R U) \simeq H_n((\lim\rightarrow M_i \otimes_R U)) \simeq \lim\rightarrow H_n(M_i \otimes_R U) \).

So \( \widetilde{\text{Tor}}_n^R(\lim\rightarrow M_i, N) \simeq \lim\rightarrow \widetilde{\text{Tor}}_n^R(M_i, N) \) (by Theorem 3). □

A similar argument gives:

**Proposition 5.** If \( R \) is Gorenstein then \( \widetilde{\text{Tor}}_n^R(M, \lim\rightarrow N_i) \simeq \lim\rightarrow \widetilde{\text{Tor}}_n^R(M, N_i) \) for any right \( R \)-module \( M \), any \( n \in \mathbb{Z} \) and any inductive system \( (N_i)_{i \in I} \) of left \( R \)-modules.

5. **Computing** \( \widetilde{\text{Tor}}_n^R(M, N) n \geq 1 \) by the combined use of a flat and a Gorenstein flat resolution of \( M \)

Let \( R \) be a commutative noetherian ring and let \( M \) be an \( R \)-module that has a Gorenstein flat resolution. If \( F \) is a deleted flat resolution, \( G \) is a deleted Gorenstein flat resolution of \( M \), and \( v : F \to G \) is a map of complexes induced by \( \text{id}_M \) then for each \( R \)-module \( N \) and for each \( n \geq 1 \) let

\[
\text{tor}_n^R(M, N) = H_{n+1}(M(v) \otimes_R N)
\]

(by [6], pp. 392, \( \text{tor}_n^R(M, -) \) is well defined).
We prove that for a commutative Gorenstein ring $R$ these are the Tate torsion functors $\widehat{\text{Tor}}_n^R(M, N)$, for $n \geq 1$.

We note first that a similar argument to the proof of Theorem 1 shows the existence of an Avramov-Martsinkovsky type exact sequence with $\text{Tor}$, $\text{gtor}$, and $\text{tor}$:

**Proposition 6.** Let $R$ be a commutative noetherian ring and let $M$ be an $R$-module that has a Gorenstein flat resolution. For each $R$-module $N$ there is an exact sequence:

$$\ldots \to \text{gtor}_2^R(M, N) \to \text{tor}_1^R(M, N) \to \text{Tor}_1^R(M, N) \to \text{gtor}_1^R(M, N) \to 0$$

We use this result to prove that:

**Proposition 7.** If $R$ is a commutative Gorenstein ring then the following are equivalent for an $R$-module $L$:

1. $\text{proj dim } L < \infty$
2. $\text{tor}_n^R(L, -) = 0$ for any $n \geq 1$.

*Proof.* 1) $\Rightarrow$ 2) $R$ is Gorenstein, so flat dim $L < \infty$. Let $F : 0 \to F_n \to \ldots \to F_1 \to F_0 \to L \to 0$ be a minimal flat resolution of $L$ (in the sense of Definition 5 when $P$ is the class of flat $R$-modules; by [2], Theorem 7.4.4 such a resolution always exists). Let $C_0 = \text{Ker}(F_0 \to L)$, $C_i = \text{Ker}(F_i \to F_{i-1})$ for $i \geq 1$. Since $C_i$ is cotorsion (by [2], Lemma 5.3.25) and flat dim $C_i < \infty$, we have $\text{Ext}_1^R(G, C_i) = 0$ for any Gorenstein flat module $G$, for any $i \geq 0$ ([2], Corollary 10.4.27). So $\text{Hom}(G, F)$ is exact for any Gorenstein flat module $G$. Thus $F$ is a Gorenstein flat resolution of $L$. Since the exact sequence of complexes $0 \to F_n \to M(id) \to F_{n-1} \to \ldots \to F_1 \to F_0 \to L \to 0$ is split exact in each degree, for each $R$-module $N$ we have an exact sequence of complexes $0 \to F_n \to M(id) \otimes_R N \to F_{n-1} \otimes_R N \to \ldots \to F_1 \otimes_R N \to F_0 \otimes_R N \to L \otimes_R N \to 0$. The associated long exact sequence: $\ldots H_{n+1}(F, [1] \otimes_R N) \to H_n(F, \otimes_R N) \to H_n(M(id) \otimes_R N) \to H_n(F, [1] \otimes_R N) \to \ldots$ gives us $H_{n+1}(M(id) \otimes_R N) = 0 \Leftrightarrow \text{tor}_n^R(L, N) = 0$ for any $n \geq 1$, for any $R$-module $N$.

2) $\Rightarrow$ 1) Since $\text{tor}_n^R(L, -) = 0$, the exact sequence $\ldots \to \text{tor}_1^R(M, N) \to \text{Tor}_1^R(L, N) \to \text{gtor}_1^R(L, N) \to 0$ gives $\text{Tor}_n^R(L, -) \simeq \text{gtor}_n^R(L, -)$. For $N$ Gorenstein flat we obtain $\text{Tor}_n^R(L, N) \simeq \text{gtor}_n^R(L, N) = 0$ for any $n \geq 1$. Since $\text{Tor}_n^R(N, L) \simeq \text{Tor}_n^R(L, N) = 0$ for any $n \geq 1$, for any Gorenstein flat $R$-module $N$ it follows that $\text{proj dim } L < \infty$ ([2], Proposition 11.5.9).
The main result of this section (Proposition 10) shows that when $R$ is commutative Gorenstein we have $\text{Tor}_n^R(M, N) \simeq \text{Tor}_n^R(M, N)$ for any $R$-modules $M, N$, for any $n \geq 1$.

The proof uses the following property of the functors $\text{Tor}_n^R(-, -)$ (Proposition 8):

if $R$ is Gorenstein then a $\text{Hom}(\text{Gor Flat}, -)$ exact sequence $0 \to M' \to M \to M'' \to 0$ gives a long exact sequence:

$$\ldots \to \text{Tor}_2^R(M'', -) \to \text{Tor}_1^R(M', -) \to \text{Tor}_1^R(M, -) \to \text{Tor}_1^R(M'', -) \to 0$$

as well as a similar result for the functors $\text{Tor}_n^R(-, -)$ (Proposition 9):

if $R$ is Gorenstein then a $\text{Hom}(\text{Gor Proj}, -)$ exact sequence $0 \to M' \to M \to M'' \to 0$ gives a long exact sequence:

$$\ldots \to \text{Tor}_2^R(M'', -) \to \text{Tor}_1^R(M', -) \to \text{Tor}_1^R(M, -) \to \text{Tor}_1^R(M'', -) \to 0$$

The proofs of Propositions 8 and 9 use the following result:

If $\mathcal{P}, \mathcal{C}$ are two precovering classes closed under finite direct sums such that $\text{Proj} \subset \mathcal{P} \subset \mathcal{C}$ and $0 \to M' \to M \to M'' \to 0$ is a $\text{Hom}(\mathcal{C}, -)$ exact sequence of $R$-modules then there is an exact sequence of complexes $0 \to M(u) \to M(\omega) \to M(v) \to 0$ which is split exact in each degree, with $u : F' \to G'$, $(\omega : F \to G$, and $v : F'' \to G''$, respectively) a map of complexes induced by $\text{id}_{M'} (\text{id}_M, \text{id}_{M''}$ respectively), where $F'$. ($F, F''$, respectively) is a deleted $\mathcal{P}$ resolution of $M' (M, M''$ respectively) and $G'$. ($G, G''$, respectively) is a deleted $\mathcal{C}$ resolution of $M' (M, M''$ respectively) ([5], proof of Proposition 1).

**Proposition 8.** If $R$ is a Gorenstein ring and $0 \to M' \to M \to M'' \to 0$ is a $\text{Hom}(\text{Gor Flat}, -)$ exact sequence of $R$-modules then for any $R$-module $N$ there is an exact sequence:

$$\ldots \to \text{Tor}_2^R(M'', N) \to \text{Tor}_1^R(M', N) \to \text{Tor}_1^R(M, N) \to \text{Tor}_1^R(M'', N) \to 0$$

**Proof.** Since $R$ is Gorenstein, $\text{Gor Flat}$ is precovering, $\text{Proj} \subset \text{Flat} \subset \text{Gor Flat}$, $\text{Flat}$ and $\text{Gor Flat}$ are closed under finite direct sums, so we have an exact sequence of complexes:

$0 \to M(u) \to M(\omega) \to M(v) \to 0$

where $u : F' \to G'$, $(\omega : F \to G$, and $v : F'' \to G''$, respectively) is a map of complexes induced by $\text{id}_{M'} (\text{id}_M, \text{id}_{M''}$ respectively), $F'$. ($F,$
$F'$, respectively) is a deleted flat resolution of $M'(M, M''$ respectively) and $G'$. ($G, G''$, respectively) is a deleted Gorenstein flat resolution of $M'(M, M''$ respectively).

The sequence $0 \to M(u) \to M(\omega) \to M(v) \to 0$ is split exact in each degree, so for each $N$ we have an exact sequence

$$0 \to M(u) \otimes_R N \to M(\omega) \otimes_R N \to M(v) \otimes_R N \to 0.$$ 

The associated long exact sequence

$$\ldots \to H_{n+1}(M(v) \otimes_R N) \to H_n(M(u) \otimes_R N) \to H_n(M(\omega) \otimes_R N) \to \ldots$$

is the sequence

$$\ldots \to \text{tor}_2(M'', N) \to \text{tor}_1(M', N) \to \text{tor}_1(M, N) \to \text{tor}_1(M'', N) \to 0$$

A similar argument shows:

**Proposition 9.** If $R$ is a Gorenstein ring and $0 \to M' \to M \to M'' \to 0$ is a $\text{Hom}(\text{Gor Proj}, -)$ exact sequence of $R$-modules then for any $R$-module $N$ there is an exact sequence:

$$\ldots \to \widehat{\text{Tor}}_2^R(M'', N) \to \widehat{\text{Tor}}_1^R(M', N) \to \widehat{\text{Tor}}_1^R(M, N) \to \widehat{\text{Tor}}_1^R(M'', N) \to 0$$

We can prove now:

**Proposition 10.** If $R$ is a commutative Gorenstein ring then $	ext{tor}_n^R(M, N) \simeq \widehat{\text{Tor}}_n^R(M, N)$ for any $R$-modules $M$ and $N$, any $n \geq 1$.

**Proof.** Let $M$ be an $R$-module.

$R$ is Gorenstein so there is an exact sequence $0 \to L \to P \to M \to 0$ with $P \to M$ a Gorenstein projective precover and with $\text{proj dim } L < \infty$ ([2], Theorem 11.5.1). By [2], Lemma 11.7.7, there is an exact sequence $0 \to L \to C \to K \to 0$ such that $K$ is flat and $C$ is cotorsion with finite projective dimension.

We consider the following pushout diagram:
$F$ is Gorenstein flat since $K$ and $P$ are.

So there is an exact sequence $0 \to C \to F \to M \to 0$ with $F$ Gorenstein flat, and $C$ cotorsion with finite projective dimension.

Since $\text{flat dim } C < \infty$ and $C$ is cotorsion the sequence is $\text{Hom}(\text{Gor Flat}, -)$ exact ([2], Corollary 10.4.27). By Proposition 8, for each $R$-module $N$ we have an exact sequence:

$$\ldots \to \text{tor}_2^R(M, N) \to \text{tor}_1^R(C, N) \to \text{tor}_1^R(F, N) \to \text{tor}_1^R(M, N) \to 0$$

By Proposition 7, $\text{tor}_n^R(C, N) = 0$ for any $n \geq 1$. So

$$\text{tor}_n^R(M, N) \simeq \text{tor}_n^R(F, N) \text{ for any } n \geq 1.$$  

Since $\text{proj dim } C < \infty$ the sequence $0 \to C \to F \to M \to 0$ is $\text{Hom}(\text{Gor Proj}, -)$ exact. So (by Proposition 9) we have an exact sequence:

$$\ldots \to \text{Tor}_2^R(M, N) \to \text{Tor}_1^R(C, N) \to \text{Tor}_1^R(F, N) \to \text{Tor}_1^R(M, N) \to 0$$
We have \( \widehat{\Tor}_n^R(C, -) = 0 \) for any \( n \geq 1 \) (by Corollary 1). It follows that
\[
(14) \quad \widehat{\Tor}_n^R(M, N) \simeq \widehat{\Tor}_n^R(F, N) \quad \text{for any } n \geq 1.
\]

\( F \) is Gorenstein flat, so \( G\tor_n^R(F, -) = 0 \) for any \( n \geq 1 \).

The long exact sequence
\[
\ldots \to G\tor_2^R(F, N) \to \widehat{\Tor}_1^R(F, N) \to \tor_1^R(F, N) \to G\tor_1^R(F, N) \to 0
\]
gives us \( \widehat{\Tor}_n^R(F, N) \simeq \tor_n^R(F, N) \) for any \( n \geq 1 \).

The exact sequence
\[
\ldots \to G\tor_2^R(F, N) \to \tor_1^R(F, N) \to \tor_1^R(F, N) \to G\tor_1^R(F, N) \to 0
\]
gives us \( \tor_n^R(F, N) \simeq \tor_n^R(F, N) \) for any \( n \geq 1 \).

So
\[
(15) \quad \widehat{\Tor}_n^R(F, N) \simeq \tor_n^R(F, N) \simeq \tor_n^R(F, N)
\]
for any left \( R \)-module \( N \), for any \( n \geq 1 \).

By (13), (14), (15) we have \( \tor_n^R(M, N) \simeq \tor_n^R(M, N) \) for any \( M, N \), for any \( n \geq 1 \).

\[\square\]

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References

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